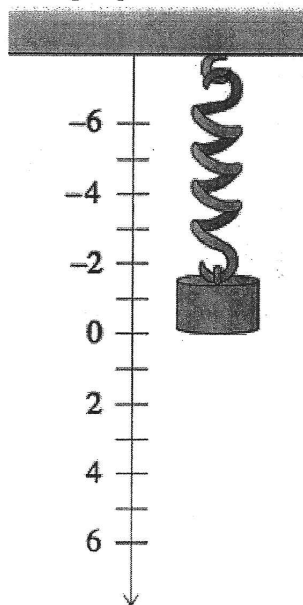


Zak Basta

Modeling the Motion of a Spring

Consider a weight attached to a spring that is suspended from a horizontal bar as illustrated in the figure. When the object comes to rest we say it is at "equilibrium" which is labeled 0 on the vertical number line. If you give the weight a push, either up or down, it will start to move and the motion can be modeled by sine and cosine functions. The "stiffness" of the spring and the mass of the object affect how far the object moves from the equilibrium position. The initial velocity and initial position also affect the motion of the spring. (We don't always start at the equilibrium position.)



If we neglect any damping forces (air resistance etc.) then the motion of the spring can be modeled by

$$x(t) = \frac{v_0}{\omega} \sin(\omega t) + x_0 \cos(\omega t)$$

where $x(t)$ is the position of the object along the number line at time t . The other quantities are constants: ω is a constant that depends on the stiffness of the spring and the mass of the weight, v_0 is the initial velocity, and x_0 is the initial position of the object.

Model the motion of a weight on a spring:

Suppose a weight is set in motion from a position 3 centimeters below the equilibrium position and with a downward velocity of 4 centimeters per second. (Please note that the vertical number line used for position is "upside down". This is a convention from physics and it means that positions below equilibrium actually correspond to a positive value.) Assume that the spring stiffness and mass of the weight mean that $\omega = 2$ for this system.

Part I

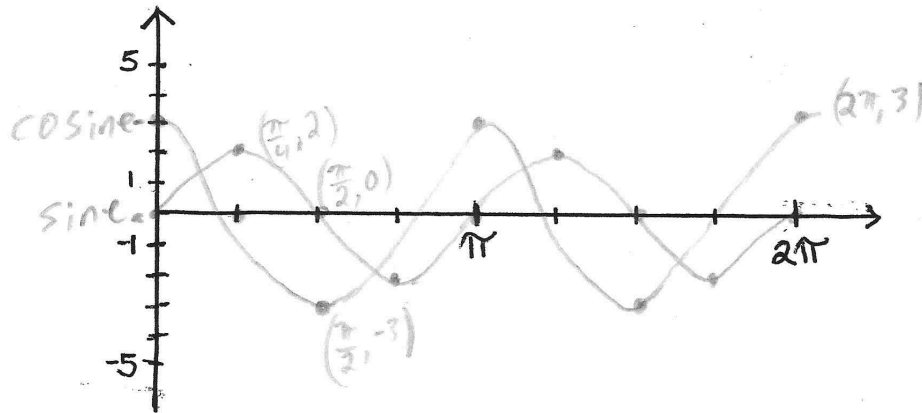
1) Write the function $x(t)$ that gives the position of the weight as a function of time t in seconds. (Your function should consist of a sine term and a cosine term.)

$$x(t) = \frac{4 \text{ cm/sec}}{2} \sin(2t) + 3 \cos(2t)$$

$$x(t) = 2 \text{ cm/sec} \cdot \sin(2t) + 3 \cos(2t)$$

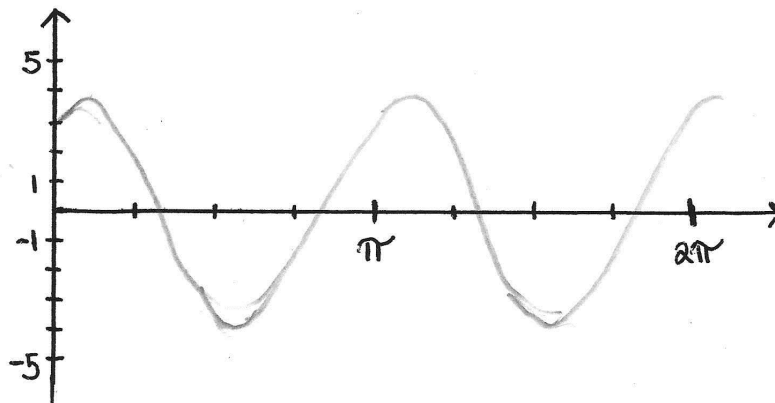
$$x(t) = 2 \sin(2t) + 3 \cos(2t)$$

- 2) Graph the separate sine and cosine components of your function from (1) on the same set of axes. That is graph $x_1 = \frac{v_0}{\omega} \sin(\omega t)$ and $x_2 = x_0 \cos(\omega t)$ on the set of axes below. (Sketch these graphs by hand and show two full cycles.)



- 3) Use a graphing calculator (or online graphing utility) to graph the entire function from part (1). Use the window settings indicated below. Sketch what you see on your calculator display.

xmin = 0
 xmax = 2π
 xscl = $\frac{\pi}{4}$
 ymin = -5
 ymax = 5
 yscl = 1



- 4) Write an equation for your calculator graph in the form $x(t) = A \cos[B(t-C)]$. (Use the trace or maximum feature of your graphing utility to help you find values for A, B, and C. I expect to see decimal approximations for these values.)

$A = 3.5$ $B = 2$
 $C = 3$

$$x(t) = 3.5 \cos[2(t-3)]$$

5) How are the graphs from part 2) related to the graph in part 3)? Are the values for period and amplitude the same or different? Why do you think we see these results? Please write out your explanation using complete sentences.

- After putting the two graphs together, just like in vectors, we get the sum of $A \sin \phi$ at each point
- The period remains the same as nothing makes the graph complete a wave any faster.
- The amplitude changes with the value of A .

Part II:

6) Prove that the following is an identity (A is a positive constant) by filling in the blanks below.

$$A \sin \phi \sin(\omega t) + A \cos \phi \cos(\omega t) = A \cos(\omega t - \phi)$$

$A \sin \phi \sin(\omega t) + A \cos \phi \cos(\omega t)$	$= A(\sin \phi \sin(\omega t) + \cos \phi \cos(\omega t))$	Factor out A
	$= A(\cos \phi \cos(\omega t) + \sin \phi \sin(\omega t))$	Commutative Property of Addition
	$= A[\cos(\omega t) \cos \phi + \sin(\omega t) \sin \phi]$	Commutative Property of Multiplication
	$= A \cos(\omega t - \phi)$	Difference identity for cosine

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

Part III:

Rewrite an expression of the form

① $c_1 \sin(\omega t) + c_2 \cos(\omega t) = A \sin \phi \sin(\omega t) + A \cos \phi \cos(\omega t)$ in terms of a cosine function:

② $A \cos(\omega t - \phi)$. Use the following **definitions** and your result from part II:

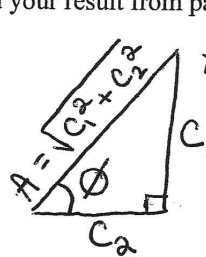
① $c_1 = A \sin \phi \Rightarrow \sin \phi = \frac{c_1}{A}$

② $c_2 = A \cos \phi \Rightarrow \cos \phi = \frac{c_2}{A}$

$\tan \phi = \frac{c_1}{c_2}$

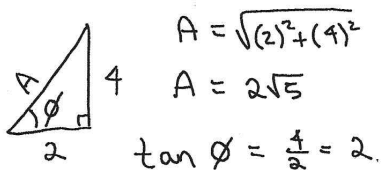
$$A = \sqrt{c_1^2 + c_2^2}$$

A , is the amplitude of the cosine function and ϕ , "phi", is called the phase angle and is measured in radians.



"Visual Reminder"
Please note that c_1, c_2 may possibly be negative.

Example: Write $4 \sin 3t + 2 \cos 3t$ in terms of a cosine function.



$$A = \sqrt{(2)^2 + (4)^2}$$

$$A = 2\sqrt{5}$$

$$\tan \phi = \frac{4}{2} = 2$$

$$\phi = \tan^{-1}(2) \approx 1.12$$

↑
! Radian mode !

$$4 \sin 3t + 2 \cos 3t \approx \boxed{2\sqrt{5} \cos(3t - 1.12)}$$

$$\approx 2\sqrt{5} \cos[3(t - 0.37)]$$

7) Rewrite your function from part 1) in terms of a cosine function: $A \cos(\omega t - \phi)$. Show your work.

$$A^2 = 2^2 + 3^2$$

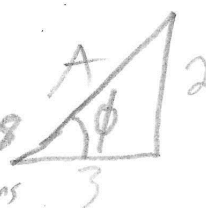
$$A^2 = 13$$

$$A = \pm \sqrt{13}$$

$$\tan \phi = \frac{2}{3}$$

$$\phi = \tan^{-1}\left(\frac{2}{3}\right) \approx .588$$

radians



$$X(t) = 2 \sin 2t + 3 \cos 2t$$

$$\approx \sqrt{13} \cos(2t - .588)$$

$$\approx \sqrt{13} \cos(2t - .294)$$

8) Compare your function from part 4) and part 7). What do you observe? Write a one or two sentence explanation.

They are nearly the same.

$3.6^2 = 12.96$, so $3.5 \approx \sqrt{13}$. $\tan^{-1}(2/3)$ in radian mode $\approx .588$ and $\frac{\pi}{6} \approx .524$. Very similar.

4) $3.5 \cos[2(t - .3)]$

7) $\sqrt{13} \cos(2t - .294)$